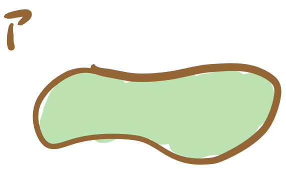
MATH 6021 Lecture 6 10/12/2020

Recall: Plateau's Problem: Given simple closed curve $\Gamma \subseteq \mathbb{R}^3$,

$\exists$ minimal "surface" Σ with $\partial\Sigma = \Gamma$ which is area-minimizing?

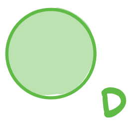


Douglas-Rado Thm:

Any such Γ bounds an area-minimizing disk.

$\uparrow u: D \to \mathbb{R}^3$

$u = (u_1, u_2, u_3)$



$$\text{Area}(u) := \int_D \sqrt{|u_x|^2|u_y|^2 - \langle u_x, u_y\rangle^2}\, dx\,dy$$

$$\text{Energy}(u) := \frac{1}{2}\int_D |\nabla u|^2\, dx\,dy$$

$X_\Gamma = \{\text{all such } u\}$

Last time: $\inf_{u\in X_\Gamma} \text{Area}(u) = \inf_{u \in X_\Gamma} \text{Energy}(u)$.

Proof of Douglas-Rado Thm

Step 1: Given any $u|_{\partial D} : \partial D \to \Gamma$, monotone, onto, we can find an energy-minimizing harmonic map $u: D \to \mathbb{R}^3$ with prescribed boundary value.

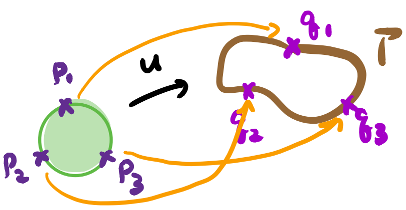
Note: Solving the Dirichlet problem for $\Delta u^i = 0$, $u = (u^1, u^2, u^3)$ which is smooth in D.

Step 2: Minimize among "all" boundary parametrizations.

Ingredient A: "Normalization" of boundary parametrizations.

(Recall: Conf(D) is non-cpt)

3-point Lemma: Consider $\mathcal{F} := \left\{ u : D \to \mathbb{R}^3 \;\middle|\; \begin{array}{l} u \text{ satisfies (1), (2), \& } \text{Energy}(u) \le \frac{K}{2} \\ \text{s.t. } u(p_i) = q_i \quad i = 1,2,3 \end{array} \right\}$



THEN, $\{ u|_{\partial D} : \partial D \to \Gamma \mid u \in \mathcal{F}\}$ is an equi-cts family.

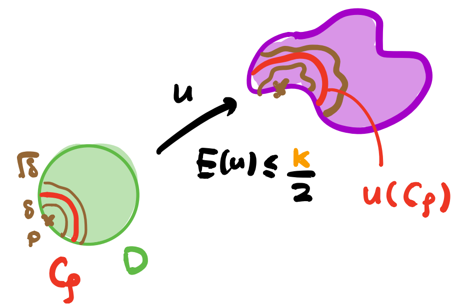
(Ref: C.M. Lemma 4.1 & 4.14)

Ingredient B: Courant-Lebesgue Lemma

Suppose $u: D \to \mathbb{R}^3$, $u \in C^0(\bar{D}) \cap W^{1,2}(D)$. Energy$(u) \leq \frac{K}{2}$.

Then, $\forall \delta \in (0,1)$, $\exists \rho \in [\delta, \sqrt{\delta}]$ s.t.

$$\text{Length}(u(C_\rho))^2 \leq \frac{4\pi K}{-\log \delta} \xrightarrow{\delta \searrow 0} 0$$



Finally, if $\{u_j\}$ is a minimizing seq. of harmonic maps on D satisfying the 3-point condition, then

$\{u_j|_{\partial D}\}$ equi-cts $\xRightarrow[\text{Ascoli}]{\text{Azela-}}$ $\exists$ subseq. uniform limit $u_j \xrightarrow{C^0} u_\infty$ on ∂D $\xRightarrow[\text{principle}]{\text{max}}$ $u_\infty : D \to \mathbb{R}^3$ harmonic is the desired energy minimizer.

$\square$

Remaining Question: Existence of branch points?

Remark: Gulliver '73 : $\nexists$ interior branch pts (ie. immersion)

Open Question: $\exists$ boundary branch points?

Remark: The mapping approach can be used to construct minimal spheres or incompressible minimal surfaces in Riemannian manifolds (c.f. Sacks-Uhlenbeck '81, Schoen-Yau '79)

Some Drawbacks

- these min. surfaces only be immersed, not nec. embedded (c.f. Meeks-Yau 1980's)
- method only works for 2-dim surfaces (c.f. B. White ~1980's)
- need to restrict the topology of the min. surface "a-priori"

Eg.)

Q: Is there a more general approach that works in any dimension / codimension & in Riem. setting?

E.g.) Consider the following:

Let $\Gamma \subseteq \mathbb{R}^n$ be a closed $(k-1)$-dim. embedded submanifold.

Problem: Find "k-dim. surfaces" $\Sigma \subseteq \mathbb{R}^n$ with $\partial\Sigma = \Gamma$ s.t.

$$|\Sigma| = \inf\{ |\tilde{\Sigma}| : \partial\tilde{\Sigma} = \Gamma \} =: \alpha .$$

"Direct Method": Fix a min. seq. $\{\Sigma_i\}$ s.t. $\lim_i |\Sigma_i| = \alpha$

Q1: (Compactness) $\exists$ subseq. $\Sigma_{i'} \xrightarrow{?} \Sigma$ in some sense? $\partial\Sigma = \Gamma$?

Q2: (Lower semi-continuity) $|\Sigma| \leq \lim_{i \to \infty} |\Sigma_{i'}|$

Key: What class of "surfaces" $\tilde{\Sigma}$ are we looking at?

One Answer: Integral Currents by de Rham, Federer-Fleming.

Overview of Integral Currents

[Ref: L. Simon "Lectures on Geometric Measure Theory"]

Key Idea: Want a notion of "surfaces" which allow "singularities", "multiplicities", "orientation"

The desired objects are "currents": (in $\mathbb{R}^n$)

$$T = (S, \Theta, \xi)$$

k-dimensional integer-rectifiable current

sub set "spt T"

multiplicity

orientation

where • $S \subseteq \mathbb{R}^n$ is a "k-rectifiable set", i.e. $\mathcal{H}^k(S) < +\infty$ (Hausdorff measure)

and $\exists$ bad points $E \subseteq S$ s.t. $\mathcal{H}^k(E) = 0$ and

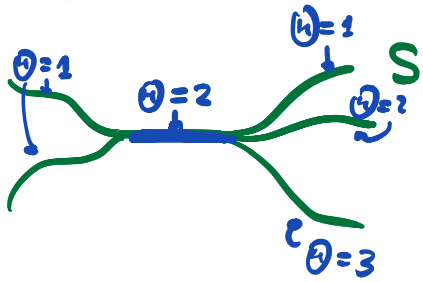
$$S \setminus E \subseteq \bigcup_{\alpha=1}^{\infty} M_\alpha^k \quad \text{where } M_\alpha^k : \text{k-dim } C^1\text{-embedded submanifold}$$

E.g.)

Fact: S has a tangent space $T_x S$ defined a.e.

• $\Theta : S \to \mathbb{Z}_{\geq 0}$ non-negative, $\mathbb{Z}$-valued, $\mathcal{H}^k$-measurable function

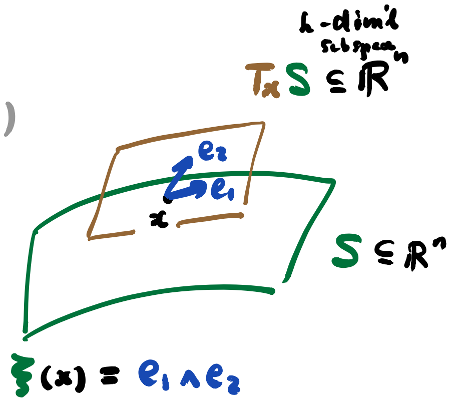
E.g.)



• $\xi : S \longrightarrow \left\{ \begin{matrix} \text{unit} \\ \text{simple} \\ \text{k-vector} \end{matrix} \right\} \subseteq \Lambda^k \mathbb{R}^n$ s.t.

$\xi(x)$ assigns an "orientation" of $T_x S$ a.e.

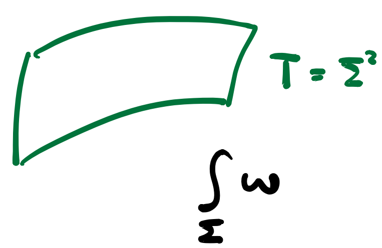
E.g.)



Key Property 1: We can "integrate" smooth k-forms $\omega \in \Omega_c^k(\mathbb{R}^n)$ on a k-dim. rectifiable current T in $\mathbb{R}^n$.

ω : 2-form in $\mathbb{R}^3$

$$T(\omega) := \int_S \omega_x(\xi_x) \underbrace{\Theta(x)\, d\mathcal{H}^k(x)}_{d\mu(x)}$$



Note: T is a linear functional on $\Omega_c^k(\mathbb{R}^n)$

$\rightsquigarrow$ "Weak topology" i.e. $T_i \to T$ iff $T_i(\omega) \to T(\omega)$ $\forall \omega \in \Omega_c^k(\mathbb{R}^n)$.

Define: $\mathbb{M}(T) := \sup\{ T(\omega) : \omega \in \Omega^k_c(\mathbb{R}^n),\ |\omega|_\infty \leq 1 \}$

mass of T (= area counting multiplicities)

Locally, $\mathbb{M}_W(T)$ = mass of T inside $W \subset \mathbb{R}^n$.

Key Property 2: $\exists$ a notion of "boundary" for currents

$$\partial : \{k\text{-current}\} \longrightarrow \{(k-1)\text{-current}\}$$

$$(\partial T)(\eta) := T(d\eta) \qquad \forall\, \eta \in \Omega^{k-1}_c(\mathbb{R}^n)$$

i.e. "Stokes' Thm" hold for currents $\left(\int_\Sigma d\eta = \int_{\partial\Sigma} \eta \right)$

Remark: $d^2 = 0$ for forms $\Rightarrow$ $\partial^2 = 0$ for currents

Def^{n}: A k-current T is said to be a "cycle" if $\partial T = 0$

Definition: If T & ∂T are integer-rectifiable currents then we say that T is an "integral current".

Some Useful Theorems for Integral Currents:

Compactness Theorem: Let $\{T_j\}$ be a seq. of k-dim integral currents

s.t. $$\sup_j \left(\mathbb{M}_W(T_j) + \mathbb{M}_W(\partial T_j) \right) < +\infty \qquad \forall\, W \subset\subset \mathbb{R}^n.$$

THEN, $\exists$ k-dim integral current T and a subseq. $T_{j'} \longrightarrow T$

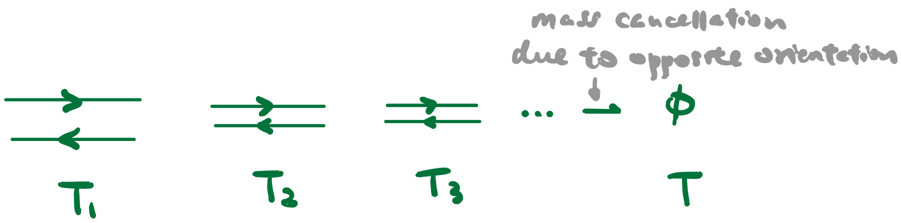
[Actually. $\partial T_{j'} \longrightarrow \partial T$ since ∂ is cts w.r.t weak topology.]

Lower Semi-continuity Theorem:

$$T_j \to T \implies \mathbb{M}_W(T) \le \varliminf_{j\to\infty} \mathbb{M}_W(T_j) \quad \forall\, W \subset\subset \mathbb{R}^n$$

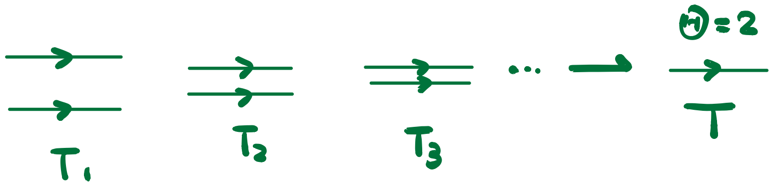
Note: Mass can jump down in the "week" limit.

Eg. 1)



← mass drop

Eg. 2)



← mass cts

Homology Isomorphism Theorem: The boundary operator ∂ on integral currents (recall: $\partial^2 = 0$) defines a homology theory isomorphic to the singular homology with $\mathbb{Z}$-coefficients.

Note: Same also holds for other coefficient e.g. $\mathbb{Z}_2$.

Federer-Fleming Thm:

Given any $(k-1)$-dim smooth, closed, embedded submfd $\Gamma \subseteq \mathbb{R}^n$, then $\exists$ k-dim integral current T in $\mathbb{R}^n$ which minimizes mass among all k-dim. integral currents T' with $\partial T' = \Gamma$.

as $(k-1)$-current

Q: How "regular" / "smooth" is the minimizer T ?